Math 211 - Bonus Exercise 8 (please discuss on Forum)

1) For any group G, define normal subgroups of G inductively by $Z_0 = \{1\}$ and for all $i \geq 0$ setting

$$Z_i \trianglelefteq Z_{i+1} \trianglelefteq G \quad \text{such that} \quad Z_{i+1}/Z_i = \text{center of } G/Z_i$$

(in other words, once Z_i is defined, Z_{i+1} is the subgroup of G which the Correspondence Theorem matches with the center of G/Z_i as a normal subgroup of G/Z_i. Show that G is nilpotent if and only if the series

$$\{1\} = Z_0 \trianglelefteq Z_1 \trianglelefteq Z_2 \trianglelefteq \dots \tag{1}$$

eventually terminates with $Z_k = G$ for some k. The series (1) is called the **upper central series of** G.

2) Prove that the following statements are equivalent:

a) any nonabelian finite simple group has even order

b) the only simple groups of odd order are $\mathbb{Z}/p\mathbb{Z}$

c) every group of odd order is solvable

(the fact that the statements above are true is a very difficult result due to Feit-Thompson).

3) Let G be the group of unitriangular $n \times n$ matrices (with coefficients in any field)

$$G = \left\{ \begin{pmatrix} 1 & a_{12} & \vdots & a_{1,n-1} & a_{1n} \\ 0 & 1 & \vdots & a_{2,n-1} & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \vdots & 1 & a_{n-1,n} \\ 0 & 0 & \vdots & 0 & 1 \end{pmatrix}, a_{ij} \text{ anything} \right\}$$

with respect to multiplication. Compute the subgroups $G^{\{i\}} = [G^{\{i-1\}}, G]$ for all i and show that G is a nilpotent group. Also compute the upper central series, as defined above.

4) Compute the upper and lower central series of D_6 and D_8. Which of them is nilpotent?